Exercises in Nonholonomic Mechanics

Exercise 1. *Consider the Lagrangian*

$$L = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - mgz$$

subject to the constraint

$$y\dot{x} - x\dot{y} = 0$$

1. *Are these constraints holonomic or nonholonomic?*
2. *Write down the dynamic nonholonomic equations.*
3. *Write down the variational nonholonomic equations. Are they the same?*

Exercise 2. *Consider the Chaplygin sleigh. The Lagrangian is*

$$L = \frac{1}{2}\left(m\dot{x}^2 + m\dot{y}^2 + (I + ma^2)\dot{\theta}^2 + 2ma\dot{\theta}\left(\dot{y}\cos\theta - \dot{x}\sin\theta\right)\right).$$

The knife edge constraint (no perpendicular motion) is

$$\dot{y}\cos\theta - \dot{x}\sin\theta = 0.$$

1. *Is this system holonomic or nonholonomic?*
2. *Find the dynamic nonholonomic equations. In particular, find the dynamics of v and ω where*

$$v = \dot{x}\cos\theta + \dot{y}\sin\theta, \quad \omega = \dot{\theta}.$$

3. *Conclude that as time goes to infinity, $\omega \to 0$ and v approaches a positive number.*

Exercise 3. *Show that if one has a conserved quantity for a holonomic mechanical system and one treats it as a nonholonomic constraint using the Lagrange-d'Alembert principle, then one gets the correct holonomic equations of motion (restricted to the surface of the constant conserved quantity).*

Exercise 4. *Let $\mathfrak{g}$ be a Lie algebra with $\omega = \sum_i \omega^i e_i \in \mathfrak{g}$ where $\{e_i\}$ is a basis. Consider the kinetic energy Lagrangian*

$$L = \frac{1}{2}\langle\omega, I\omega\rangle = \frac{1}{2}\sum_{ij} I_{ij}\omega^i\omega^j.$$

1. *Show that the momenta is* $p_j = \sum_i I_{ij}\omega^i$ *and the resulting equations of motion are*

$$\dot{p}_j = \sum_{ik\ell} c_{ij}^k I^{i\ell} p_k p_\ell \qquad (\dagger)$$

2. *Let* $G = \mathrm{SO}_3$*. Show that* $(\dagger)$ *produces the standard Euler equations*

$$I\dot{\omega} = (I\omega) \times \omega.$$

Suppose we impose the left-invariant constraint

$$\langle a, \omega \rangle = \sum_i a_i \omega^i = 0, \quad a = \sum_i a_i e^i \in \mathfrak{g}^*.$$

The constrained equations of motion are given by (where λ *is an unknown multiplier to satisfy the constraints)*

$$\dot{p}_j = \sum_{ik\ell} c_{ij}^k I^{i\ell} p_k p_\ell + \lambda a_j$$

3. *Show that in the case where* $G = \mathrm{SO}_3$*, the constrained equations of motion are*

$$I\dot{\omega} = (I\omega) \times \omega + \lambda a, \quad \lambda = -\frac{I^{-1}a \cdot [(I\omega) \times \omega]}{I^{-1}a \cdot a}.$$

4. *Conclude that the constrained equations of motion are volume-preserving if there exists* $\mu \in \mathbb{R}$ *such that* $(I^{-1}a) \times a = \mu a$*.*

Exercise 5. *Consider the vertical rolling disk. The configuration space is* $Q = S^1 \times S^1 \times \mathbb{R}^2$*. Let the Lagrangian be the kinetic energy*

$$L = \frac{1}{2} I \dot{\theta}^2 + \frac{1}{2} J \dot{\varphi}^2 + \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 \right).$$

The constraints of rolling without slipping are

$$\dot{x} = R\dot{\theta} \cos\varphi, \quad \dot{y} = R\dot{\theta} \sin\varphi.$$

View this system as a bundle $\pi : Q \to S^1 \times S^1$ *with* $\pi(\theta, \varphi, x, y) = (\theta, \varphi)$ *with horizontal space*

$$\mathcal{H} = \left\{ (\dot{\theta}, \dot{\varphi}, R\dot{\theta}\cos\varphi, R\dot{\theta}\sin\varphi) \right\} \subset TQ.$$

Compute the curvature of this connection. Provide a physical intuition behind your answer.

Exercise 6. *Let* $\Psi : TQ \to \mathbb{R}$ *be a constraint function; the dynamics are constrained to the set* $M = \Psi^{-1}(0)$*. The constraint is considered ideal if*

$$\sum_i \dot{q}^i \frac{\partial \Psi}{\partial \dot{q}^i} \bigg|_M = 0$$

1. *Prove that systems with ideal constraints are energy-preserving.*
2. *Prove that constraints that are linear in velocity are ideal.*
3. *Consider the rolling ball on a turntable. The Lagrangian is the kinetic energy,*

$$L = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2 + k^2\left(\dot{\theta}^2 + \dot{\varphi}^2 + \dot{\psi}^2 + 2\dot{\varphi}\dot{\psi}\cos\theta\right)\right),$$

while the constraints are

$$\begin{aligned} \dot{x} - r\dot{\theta}\sin\psi + r\dot{\varphi}\sin\theta\cos\psi &= -\Omega y \\ \dot{y} + r\dot{\theta}\cos\psi + r\dot{\varphi}\sin\theta\sin\psi &= \Omega x \end{aligned}$$

where Ω is the angular velocity of the table. Show that this system is not energy-preserving and explain why that is reasonable.

Exercise 7. *Consider the Poisson manifold $P = \mathfrak{se}_3^*$ with the induced Lie-Poisson bracket. Let an element of $\mathfrak{se}_3$ be denoted as*

$$\xi = \begin{bmatrix} 0 & -\omega_z & \omega_y & u \\ \omega_z & 0 & -\omega_x & v \\ -\omega_y & \omega_x & 0 & w \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and we choose a basis such that

$$\xi = \omega_x e_1 + \omega_y e_2 + \omega_z e_3 + u e_4 + v e_5 + w e_6.$$

1. *Explain the constraints $\mathcal{D} = \ker e_1^* \cap \ker e_5^* \cap \ker e_6^*$.*
2. *Find the Euler-Poincaré-Suslov equations with the inertia tensor*

$$\mathcal{I} = \begin{bmatrix} I_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & I_2 + ma^2 & 0 & 0 & 0 & -ma \\ 0 & 0 & I_3 + ma^2 & 0 & ma & 0 \\ 0 & 0 & 0 & m & 0 & 0 \\ 0 & 0 & ma & 0 & m & 0 \\ 0 & -ma & 0 & 0 & 0 & m \end{bmatrix}.$$

3. *Qualitatively describe the dynamics.*