Exercises in Hybrid Dynamics

Exercise 1. *Consider the hybrid dynamical system*

$$\begin{cases} \dot{x} = f(x), & x \notin S, \\ x^+ = \Delta(x^-), & x \in S. \end{cases}$$

Suppose that $x_0 \notin S$ and there exists a time $T_0 > 0$ such that $\varphi_{T_0}(x_0) \in S$ (the trajectory impacts S in time T_0).

1. *Prove that if the flow intersects S transversely at $\varphi_{T_0}(x_0)$ then there exists an $\varepsilon > 0$ and a smooth function $\tau : \mathcal{B}_\varepsilon(x_0) \to \mathbb{R}^+$ such that for all $x \in \mathcal{B}_\varepsilon(x_0)$, $\varphi_{\tau(x)}(x) \in S$. This shows that the return map is a smooth function. Hint: Implicit function theorem.*

2. *Find an example where the flow does not intersect S transversely and the previous result fails.*

Exercise 2. *Consider the following planar hybrid dynamical system. Let the continuous dynamics be governed by (in polar coordinates)*

$$\dot{r} = 1 - r, \quad \dot{\theta} = 1.$$

Suppose that impacts occur along a ray emanating from the origin, $S = \{(r,\theta) : \theta = \alpha\}$ and let the impact map be

$$\Delta(r,\alpha) = (\beta r, \gamma),$$

where β, α, and γ are all constant parameters.

1. *Let $\beta e^{\gamma-\alpha} < 1$. Find the periodic trajectory.*

2. *By explicitly solving the system, find an analytical formula for the return map and calculate the Floquet multiplier for the orbit. Is it stable?*

3. *Find the Floquet multiplier via the hybrid variational equation to reach the same conclusion (without solving the dynamics explicitly).*

Exercise 3. *Let $x = (x_1, x_2) \in \mathbb{R}^2$ and consider the switched system*

$$\begin{aligned} \dot{x} &= Ax, \quad x \in Q_1 \cup Q_3 \\ \dot{x} &= Bx, \quad x \in Q_2 \cup Q_4 \end{aligned} \tag{$\star$}$$

where Q_i denotes the i^{th} quadrant, see Fig. 1.

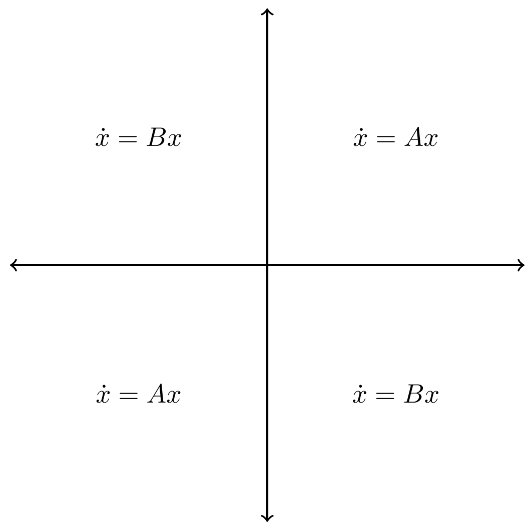

Figure 1: Portrait of the switched system in Exercise 3.

1. *Let the matrices be given by*

$$A = \begin{bmatrix} -1 & 1/5 \\ -5 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 5 \\ -1/5 & -1 \end{bmatrix}.$$

Show that the origin is stable for each system, $\dot{x} = Ax$ and $\dot{x} = Bx$, but the origin is unstable in the combined system ($\star$).

2. *Repeat the same procedure with*

$$A = \begin{bmatrix} 1 & 1/5 \\ -5 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 5 \\ -1/5 & 1 \end{bmatrix}.$$

Exercise 4. *Let G be a Lie group with $h : G \to \mathbb{R}$ with zero as a regular value and set $S = h^{-1}(0)$. When $g \notin S$, the free motion is given by the usual Euler-Poincaré equations*

$$\begin{aligned} \dot{g} &= (\ell_g)_* \, \xi, \\ \frac{d}{dt}\frac{\partial \ell}{\partial \xi} &= \mathrm{ad}_\xi^* \frac{\partial \ell}{\partial \xi}. \end{aligned}$$

1. *Write down the variational impact equations when $g \in S$.*
2. *Consider the case of $G = \mathrm{SO}_3$. Is the angular momentum Casimir preserved across impacts? Does this depend on h?*
3. *What happens if we impose the constraint $\langle a, \xi \rangle = 0$?*

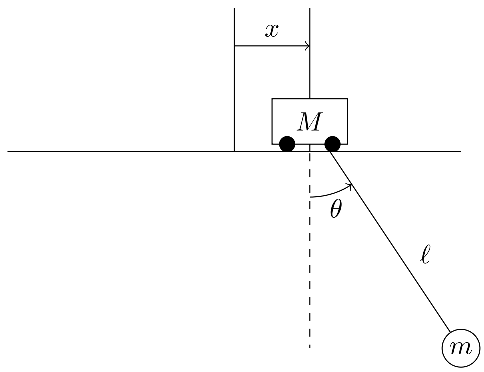

Figure 2: The pendulum on the cart.

Exercise 5. *Consider a pendulum on a cart. Its schematic is shown in Fig. 2. The Lagrangian is*

$$L = \frac{1}{2}(M+m)\dot{x}^2 + \frac{1}{2}m\left(\ell^2\dot{\theta}^2 + \ell\dot{\theta}\dot{x}\cos\theta\right) + \ell g\cos\theta,$$

where M is the mass of the cart, m is the mass of the bob, g is the acceleration due to gravity, θ is the angle the arm makes with the vertical, and x is the horizontal position of the cart.

1. *The configuration space for this case is $Q = S^1 \times \mathbb{R}$. Let $G_1 = S^1$ and $G_2 = \mathbb{R}$ act on Q via*

$$\begin{aligned} (\theta, x) &\mapsto (\theta + \alpha \bmod 2\pi, x), \quad \alpha \in S^1 \\ (\theta, x) &\mapsto (\theta, x + y), \quad y \in \mathbb{R} \end{aligned}$$

   Which group action is the system invariant under? Write down the resulting momentum equation.

2. *Suppose the cart experiences "external" impacts when $x = \pm z$ for some specified locations z. Write down the variational impact conditions. Is the momentum map found above preserved across impacts?*

3. *Suppose the cart experiences "internal" impacts when $\theta = \pm\varphi$ for some specified location φ. Write down the variational impact conditions. Is the momentum map found above preserved across impacts?*

4. *What happens if the impact location is allowed to move? i.e. repeat the above steps where $z(t)$ and $\varphi(t)$ are allowed to be time-varying.*

Exercise 6. *Consider the simple bouncing ball on a table,*

$$\begin{cases} \ddot{x} = -g, & x > 0 \\ \dot{x}^+ = -\dot{x}^-, & x = 0. \end{cases}$$

Let $\Delta(x, v) = (x, -v)$ be the impact map and let $\mu = dx \wedge dv$ be a volume on $\mathbb{R}^2$. Show that the naive calculation $\Delta^*\mu = -\mu$ implies that impacts are volume reversing. Do the correct calculation to show that volume is preserved across impacts.

Exercise 7. *Is it true that all mechanical impact systems with dissipation are Zeno?*