Exercises in Geometric Mechanics

Exercise 1. *Consider a mass in the plane $\mathbb{R}^2$ under the influence of gravity. Its Lagrangian is given by*

$$L = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) - mgy.$$

1. *Find the Lagrangian in polar coordinates: $x = r\cos\theta$, $y = r\sin\theta$.*
2. *The planar pendulum is a planar particle constrained to be a fixed distance from the origin: $r = R$. Apply this constraint to the polar Lagrangian found above to obtain a function $L_{pen} = L_{pen}(\theta, \dot{\theta})$.*
3. *Apply the Euler-Lagrange equations to recover the usual pendulum equations.*
4. *Apply the Legendre transform to L_{pen} to obtain the corresponding Hamiltonian H_{pen}. What is p_θ? Find Hamilton's equations of motion and confirm that they agree with the results from the previous part.*

Exercise 2. *To extend the previous example to three dimensions, consider the Lagrangian for the particle in $\mathbb{R}^3$ subject to gravity*

$$L = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - mgz.$$

1. *Find the Lagrangian in spherical coordinates*

$$x = r\cos\varphi\sin\theta, \quad y = r\sin\varphi\sin\theta, \quad z = r\cos\theta.$$

2. *Apply the constraint $r = R$ and write down the Lagrangian for the spherical pendulum, $L_{pen} = L_{pen}(\varphi, \theta, \dot{\varphi}, \dot{\theta})$.*
3. *Apply the Euler-Lagrange equations to find the equations of motion for the spherical pendulum.*
4. *Apply the Legendre transform to L_{pen} to obtain H_{pen}. What are the momenta p_φ and p_θ? Are either conserved? Provide both a mathematical and physical reasoning.*
5. *By turning off the influence of gravity (set $g = 0$), argue that the equations of motion (either Lagrange or Hamilton) produce the equations for great circles on a sphere.*

Exercise 3. *Consider a Lagrangian of the form*

$$L = \frac{1}{2}\sum_{i,j=1}^{n} g_{ij}(q)\dot{q}^i\dot{q}^j,$$

where $M = (g_{ij})$ is a symmetric matrix (the entries are smooth functions of position q). Show that Lagrange's equations of motion are

$$\sum_s g_{rs}\ddot{q}^s + \sum_{i,j=1}^{n} \Gamma_{rij}\dot{q}^i\dot{q}^j = 0, \quad s = 1, \dots, n$$

for some functions Γ_{rij}. Find these functions and verify that energy is conserved.

Exercise 4. *Consider the first quadrant in $\mathbb{R}^2$:*

$$Q = \{(x, y) \in \mathbb{R}^2 : x, y > 0\}.$$

1. *Show that the following is a symplectic form on Q*

$$\omega = \frac{1}{x}dx \wedge dp_x + \frac{1}{y}dy \wedge dp_y.$$

2. *Write out the formula for the Poisson bracket with respect to this symplectic form.*

3. *Consider the Hamiltonian corresponding to the "free" particle*

$$H = \frac{1}{2}\left(p_x^2 + p_y^2\right).$$

Find the equations of motion and draw some of the base integral curves.

Exercise 5. *The Lagrangian for the relativistic particle is given by*

$$L = -m_0 c\sqrt{c^2 - v^2}.$$

Use the fiber derivative to find the momenta and the energy. Find the first few terms in the Taylor expansion of the energy with respect to the velocity.

Exercise 6. *Consider a particle of mass m and electric charge e in an electromagnetic field with electric field $E = (E_x, E_y, E_z)$ and magnetic field $B = (B_x, B_y, B_z)$. The equations of motion for the particle are*

$$\frac{dp}{dt} = eE + \frac{e}{c}v \times B. \tag{†}$$

Show that (†) is Hamiltonian with

$$H = \frac{1}{2m}\left(p_x^2 + p_y^2 + p_z^2\right) + e\phi, \quad E = -\nabla\phi,$$

with the symplectic form

$$\omega = \omega_0 - \frac{e}{c}\omega_B$$
$$\omega_0 = dx \wedge dp_x + dy \wedge dp_y + dz \wedge dp_z$$
$$\omega_B = B_x dy \wedge dz + B_y dz \wedge dx + B_z dx \wedge dy$$

Confirm that energy is conserved.

Exercise 7. *Do solutions to Hamilton's equations necessarily exist for all time? Find an example with finite-time blow up.*

Exercise 8. *Let $H, K : M \to \mathbb{R}$ be two Hamiltonians on the symplectic manifold (M, ω). Suppose that they have a regular energy surface in common, $\Sigma = H^{-1}(e) = K^{-1}(e)$. Prove that the integral curves of X_H and X_K are the same on Σ except possibly for a time reparametrization.*

Exercise 9. *Let $G = \mathrm{SE}_2$ be the special Euclidean group in 2 dimensions. Elements of this group have the form*

$$g = \begin{bmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{bmatrix}$$

What is the Lie-Poisson bracket on $\mathfrak{se}_2^*$? Are there any Casimirs? What about $\mathfrak{se}_3^*$?

Exercise 10. *Let $Q = S^1 \times \mathrm{SE}_2$. We will examine the momentum map arising from two different groups; let $G_1 = \mathrm{SE}_2$ and $G_2 = S^1 \times \mathbb{R}^2$. Let the actions be defined via*

$$(\theta, \varphi, x, y) \mapsto (\theta, \varphi + \alpha \operatorname{mod} 2\pi, x\cos\alpha - y\sin\alpha + a, x\sin\alpha + y\cos\alpha + b)$$
$$(\theta, \varphi, x, y) \mapsto (\theta + \beta, \varphi, x + \lambda, y + \mu),$$

where $(\alpha, a, b) \in \mathrm{SE}_2$ and $(\beta, \lambda, \mu) \in S^1 \times \mathbb{R}^2$. What are the resulting momentum maps?

Exercise 11. *The Lie-Poisson bracket for the 2-dimensional affine algebra is*

$$\{f, g\}(x, y) = -y \cdot \left[\frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}\right].$$

Consider the Hamiltonain $H = 1/2(x^2 + y^2)$. Write down the equations of motion and draw the phase portrait. What is strange here?

Exercise 12. *Let G be a Lie group and consider the Lie-Poisson structure on $P = \mathfrak{g}^*$. Let $\mathfrak{g}$ have the structure coefficients c_{ij}^k; this means that with respect to a basis $\{e_j\}$, we have*

$$[e_i, e_j] = \sum_k c_{ij}^k e_k.$$

Suppose that G is unimodular, i.e.

$$\sum_j c_{ij}^j = 0.$$

Prove that the Hamiltonian vector fields are volume-preserving.

Exercise 13. *Let $(S, I, R) \in \mathbb{R}^3$ be coordinates and consider the following bracket*

$$\{f, g\}(S, I, R) = \langle v, \nabla f \times \nabla g \rangle, \quad v = \begin{bmatrix} aI \\ 0 \\ rSI \end{bmatrix}.$$

where $a, r > 0$ are constants.

1. *Show that the equations of motion with the Hamiltonian $H = S + I + R$ are the usual SIR epidemiological model.*

2. *Show that $C = R + \frac{a}{r} \log S$ is a Casimir of this bracket.*

3. *Using the fact that H and C are constants of motion, reduce the SIR model to the 1-dimensional system*

$$\dot{S} = -rS\left(C - H + S - \frac{a}{r}\log S\right). \tag{$\star$}$$