Primer on Symplectic Geometry

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The contents here are reasonably standard and make up the underlying geometry of Hamiltonian systems; all Hamiltonian systems evolve on a symplectic manifold. The goal is for these notes to be self-contained. However, if you're feeling adventurous, some standard references are listed at the end.

1. The book by Berndt, [3], is a very short and terse introduction to this topic. However, it is a slow and dense read.

2. A reference that many people like is Arnold's book, chapter 8 in [2].

3. My personal favorite is the book by Abraham and Marsden, [1], specifically the beginning parts of chapter 3.

1 Symplectic Linear Algebra

We begin with some pure linear algebra. Throughout, V will be a finite-dimensional real vector space.

1.1 Exterior Algebra

The dual space of V is denoted by V^* and is given by

$$V^* = \mathrm{Hom}(V, \mathbb{R}) = \{\varphi : V \to \mathbb{R} \,|\, \varphi \text{ is linear}\}.$$

Choose a basis $e_1, \ldots, e_n$ of V. Every vector is uniquely decomposed into this basis,

$$v = \sum_{i=1}^{n} v^i e_i.$$

Let $\alpha^i : V \to \mathbb{R}$ be the elements of the dual space that pick out the i^{th}-coordinate, i.e. $\alpha^i(v) = v^i$. These elements have the property

$$\alpha^i(e_j) = \delta^i_j = \begin{cases} 1 & i = j, \\ 0 & i \neq j. \end{cases} \tag{1}$$

Exercise 1. *Prove that the functions $\alpha^1, \ldots, \alpha^n$ form a basis of V^* and conclude that* $\dim V = \dim V^*$.

As the dual space contains linear functions on V, a straight-forward extension if that of multi-linear functions. A function $f : V \times \ldots V \to \mathbb{R}$ is k-linear if it is linear in each argument, i.e.

$$f(\ldots, \alpha u + \beta v, \ldots) = \alpha f(\ldots, u, \ldots) + \beta f(\ldots, v, \ldots),$$

for all scalars $\alpha, \beta \in \mathbb{R}$ and vectors $u, v \in V$. A k-linear map is called a k-tensor on V and the set of these objects will be denoted by $\mathcal{T}^k(V)$.

Exercise 2. *Prove that* $\dim \mathcal{T}^k(V) = n^k$. *How can you describe a basis?*

Three very important examples of tensors are:

1. The dot product, $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ is a *symmetric* 2-tensor.
2. The determinant, $\det : V^n \to \mathbb{R}$ is an *alternating* n-tensor.
3. Let $\alpha, \beta \in V^*$ be 1-tensors. Their wedge product, $\alpha \wedge \beta : V \times V \to \mathbb{R}$ is an *alternating* 2-tensor defined by

$$\alpha \wedge \beta(u, v) = \alpha(u)\beta(v) - \alpha(v)\beta(u) = \det \begin{bmatrix} \alpha(u) & \alpha(v) \\ \beta(u) & \beta(v) \end{bmatrix}. \tag{2}$$

An alternating k-tensor has the property that

$$f(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) = (\operatorname{sgn} \sigma) \cdot f(v_1, \ldots, v_k),$$

for any permutation $\sigma \in S_k$. An alternating k-tensor will be called a k-form. The set of all k-forms on V will be denoted by $\Lambda^k(V)$. Given a k-form and an ℓ-form, we can combine them to create a $(k+\ell)$-form via the wedge product.

$$\begin{aligned} &(f \wedge g)(v_1, \ldots, v_k, v_{k+1}, \ldots v_{k+\ell}) = \\ &\quad = \frac{1}{k!\ell!} \sum_{\sigma \in S_{k+\ell}} (\operatorname{sgn} \sigma)\, f(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) g(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+\ell)}) \end{aligned}$$

Exercise 3. *Let* α, β *be two 1-forms. Verify that* $\alpha \wedge \beta$ *is given by* (2).

The wedge product has the following properties.

Proposition 1. *Let* $f \in \Lambda^k(V)$, $g \in \Lambda^\ell(V)$, *and* $h \in \Lambda^j(V)$. *Then*

1. $f \wedge g = (-1)^{k\ell} g \wedge f$,
2. $f \wedge (g \wedge h) = (f \wedge g) \wedge h$,
3. $f \wedge (g + h) = f \wedge g + f \wedge h$, *and* $(f + g) \wedge h = f \wedge h + g \wedge h$.

It turns out that all forms can be constructed by wedging together 1-forms. This has a particularly nice forms as it neatly generalizes (2).

Proposition 2. *Let $\alpha^1, \ldots, \alpha^k \in V^*$ be 1-forms. Then*

$$\left(\alpha^1 \wedge \ldots \wedge \alpha^k\right)(v_1, \ldots, v_k) = \det\left[\alpha^i(v_j)\right]. \tag{3}$$

In the case of a 3-form, (3) manifests as

$$\alpha \wedge \beta \wedge \gamma(u, v, w) = \det \begin{bmatrix} \alpha(u) & \alpha(v) & \alpha(w) \\ \beta(u) & \beta(v) & \beta(w) \\ \gamma(u) & \gamma(v) & \gamma(w) \end{bmatrix}.$$

Proposition 2 is not restrictive at all as all k-forms can be constructed in this way. To explore this, we introduce the multi-index notation

$$I = (i_1, \ldots, i_k), \quad \alpha^I = \alpha^{i_1} \wedge \ldots \wedge \alpha^{i_k}.$$

Exercise 4. *Prove the following.*

1. *$\alpha^I \neq 0$ if and only if I has no repeating terms.*
2. *α^I and α^J are linearly independent if and only if I and J are not permutations of each other.*
3. *The (linearly independent) k-forms α^I form a basis of $\Lambda^k(V)$. Conclude that*
$$\dim \Lambda^k(V) = \binom{n}{k} = \frac{n!}{(n-k)!k!}, \quad k \leq n$$
and $\Lambda^k(V) = 0$ if $k > n$.

Most of what we will be interested in are 2-forms. To make their calculations very explicit, let $V = \mathbb{R}^3$ with the standard basis,

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

The corresponding dual basis is

$$\alpha^1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \quad \alpha^2 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}, \quad \alpha^3 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}.$$

A basis for $\Lambda^2(\mathbb{R}^3)$ is

$$\alpha^1 \wedge \alpha^2, \quad \alpha^1 \wedge \alpha^3, \quad \alpha^2 \wedge \alpha^3. \tag{4}$$

Another way to represent a 2-form is by a skew-symmetric matrix; if $f \in \Lambda^2(\mathbb{R}^3)$ then there exists a skew-symmetric matrix A_f such that

$$f(u, v) = \langle u, A_f v \rangle.$$

Exercise 5. *For the three 2-forms given in* (4)*, find their corresponding skew-symmetric matrices.*

We end this with a definition of the exterior algebra. Recall that Proposition 1 states that the wedge product is a reasonable version of multiplication.

Definition 3. *Let $\Lambda(V) = \cup_k \Lambda^k(V)$. Then $(\Lambda(V), \wedge)$ is called the exterior algebra over V.*

1.2 Symplectic Form

We begin by defining a symplectic form.

Definition 4. *A symplectic form is a 2-form $\omega \in \Lambda^2(V)$ such that it is non-degenerate, i.e. if for all $v \in V$ $\omega(u, v) = 0$, then $u = 0$.*

Example 5. *In $\mathbb{R}^2$, $\alpha^1 \wedge \alpha^2$ is a symplectic form. Moreover, in $\mathbb{R}^4$, $\alpha^1 \wedge \alpha^3 + \alpha^2 \wedge \alpha^4$ is a symplectic form.*

Exercise 6. *Verify that the examples above are correct.*

Exercise 7. *Prove that V possesses a symplectic form if and only if its dimension is even.*

Let (V, ω) be a symplectic vector space (this just means that ω is a symplectic form on V). The symplectic form induces a map

$$\omega^\flat : V \to V^*$$
$$\omega^\flat(v) = \omega(v, \cdot).$$

Since the symplectic form is non-degenerate, this map is invertible and its inverse if given by $\omega^\sharp = \left(\omega^\flat\right)^{-1}$. These two maps are called the musical isomorphisms.

As symplectic vector spaces must be even-dimensional, let $\dim V = 2n$. It turns out that the following three properties are equivalent:

1. ω is non-degenerate
2. $\omega^\flat$ is invertible, and
3. $\omega^n = \omega \wedge \ldots \wedge \omega \neq 0$.

Example 6. *Let $V = \mathbb{R}^4$ have the symplectic form $\omega = \alpha^1 \wedge \alpha^3 + \alpha^2 \wedge \alpha^4$. If we write $\omega(u, v) = \langle u, Av \rangle$, the matrix has the form*

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}.$$

It is straightforward to see that A is non-degenerate as $\det A = 1$. *The musical isomorphisms are given by*

$$\omega^\flat(v) = A^\mathsf{T} u^\mathsf{T},$$
$$\omega^\sharp(\alpha) = \alpha^\mathsf{T} A^{-1}.$$

Finally, the wedge-produce is

$$\begin{aligned} \omega^2 &= \left(\alpha^1 \wedge \alpha^3 + \alpha^2 \wedge \alpha^4\right)^2 \\ &= \cancel{\alpha^1 \wedge \alpha^3 \wedge \alpha^1 \wedge \alpha^3} + \alpha^1 \wedge \alpha^3 \wedge \alpha^2 \wedge \alpha^4 + \alpha^2 \wedge \alpha^4 \wedge \alpha^1 \wedge \alpha^3 + \cancel{\alpha^2 \wedge \alpha^4 \wedge \alpha^2 \wedge \alpha^4} \\ &= -2\alpha^1 \wedge \alpha^2 \wedge \alpha^3 \wedge \alpha^4 \neq 0. \end{aligned}$$

The two terms that cancel out have a repeating index and the final result is nonzero as no terms are repeating.

It is important to point out that ω^n is a top-form, i.e. $\omega^n \in \Lambda^{\dim V}(V)$. By a previous exercise, the determinant is a top-form and the space of top-forms is 1-dimensional. Therefore, it turns out that for a symplectic form ω, $\omega^n = C \cdot \det$ for some constant C.

1.3 Subspaces

It turns out that a ($2n$-dimensional) symplectic vector space (V, ω) possesses distinguished subspaces. To examine these, we first define the orthogonal complement of a subspace $W \subset V$,

$$W^{\perp} := \{v \in V : \omega(v, w) = 0, \forall w \in W\}.$$

A similar object is the annihilator, $W^{\circ} \subset V^*$ given by

$$W^{\circ} := \{\varphi \in V^* : \varphi(w) = 0, \forall w \in W\}.$$

These two subspaces are identified via $\omega^{\flat}(W^{\perp}) = W^{\circ}$.

Exercise 8. *Prove that* $\dim W^{\perp} = \dim V - \dim W$ *and that* $\omega^{\flat}(W^{\perp}) = W^{\circ}$.

We can now define the four distinguished subspaces.

Definition 7. *Let (V, ω) be a symplectic vector space and $W \subset V$ be a subspace.*

1. *If $\omega|_W = 0$, W is called <u>isotropic</u>.*
2. *If $W^{\perp}$ is isotropic, W is called <u>coisotropic</u>.*
3. *If $\omega|_W$ is non-degenerate, W is called a <u>symplectic subspace</u>.*
4. *If W is both isotropic and coisotropic, W is called <u>Lagrangian</u>.*

Example 8. *Again, consider $V = \mathbb{R}^4$ with the symplectic form $\omega = \alpha^1 \wedge \alpha^3 + \alpha^2 \wedge \alpha^4$. Then*

1. $W = \operatorname{span}\{e_1, e_3\}$ *is symplectic,*
2. $W = \operatorname{span}\{e_1\}$ *is isotropic,*
3. $W = \operatorname{span}\{e_1, e_2, e_3\}$ *is coisotropic, and*
4. $W = \operatorname{span}\{e_1, e_2\}$ *is Lagrangian.*

Exercise 9. *Verify this example.*

It turns out that there is another (equivalent) characterization of these subspaces. Let $\dim W = k$.

$$\begin{array}{cccccc} W & \text{isotropic} & \iff & W \subset W^{\perp} & \implies & k \leq n, \\ W & \text{coisotropic} & \iff & W^{\perp} \subset W & \implies & k \geq n, \\ W & \text{Lagrangian} & \iff & W = W^{\perp} & \implies & k = n. \end{array}$$

2 Symplectic Manifolds

Let M be a smooth manifold. For each point $x \in M$, the tangent space T_xM is a vector space. It would seem natural to give M a symplectic structure by making each of its tangent spaces a symplectic manifold. However, it is slightly more nuanced than this.

A form on a vector space generalizes to a differential form on a manifold. Let $\Omega^k(M)$ be the set of differential k-forms on the manifold M, i.e. for $\alpha \in \Omega^k(M)$ and for each $x \in M$, $\alpha_x \in \Lambda^k(T_xM)$. We also require that this map changes smoothly with x.

Remark 9. *Essentially, a differential form is a k-form whose coefficients are smooth functions.*

Example 10. *If $M = \mathbb{R}^3$, a 2-form is given by*

$$\alpha = \sin z \, dx \wedge dz + x^4 \, dy \wedge dz.$$

Differentiating a k-form produces a $(k+1)$-form. Continuing from the previous example,

$$\begin{aligned} d\alpha &= \cos z \, dz \wedge dx \wedge dz + 4x^3 \, dx \wedge dy \wedge dz \\ &= 4x^3 \, dx \wedge dy \wedge dz. \end{aligned}$$

We are now able to define a symplectic manifold.

Definition 11. *The pair (M, ω) is called a symplectic manifold if $\omega \in \Omega^2(M)$ is a differential 2-form such that*

1. *for each $x \in M$, (T_xM, ω_x) is a symplectic vector space, and*
2. *$d\omega = 0$.*

Remark 12. *Since a symplectic vector space must be even dimensional, it follows that a symplectic manifold must also be even dimensional. An interesting question is: Do all even dimensional manifolds possess a symplectic form?*

The standard example of a symplectic manifold (I think this is the only case we'll ever consider) is the cotangent bundle of a manifold. Let Q be a smooth manifold with coordinates $x = (x^1, \ldots, x^n)$. Consider the induced coordinates $(x, p) = (x^1, \ldots, x^n, p_1, \ldots, p_n)$ on T^*Q. The tautological 1-form $\vartheta \in \Omega^1(T^*Q)$ is given by

$$\vartheta = \sum_i p_i \, dx^i,$$

which induces the canonical symplectic form

$$\omega = -d\vartheta = \sum_i dx^i \wedge dp_i.$$

2.1 Poisson Manifolds

A sister concept to a symplectic manifold is a Poisson manifold.

Definition 13. *A manifold P is a Poisson manifold if there exists a bracket $\{\cdot,\cdot\}: C^\infty(P)\times C^\infty(P)\to C^\infty(P)$ such that*

1. $\{f,g\} = -\{g,f\}$ *(skew symmetric)*
2. $\{f,\alpha g+\beta h\} = \alpha\{f,g\}+\beta\{f,h\}$ *for* $\alpha,\beta\in\mathbb{R}$ *(bi-linear)*
3. $\{f,gh\} = \{f,g\}h+\{f,h\}g$ *(Liebniz' rule)*
4. $\{f,\{g,h\}\}+\{g,\{h,f\}\}+\{h,\{f,g\}\} = 0$ *(Jacobi identity)*

The first two conditions seem very similar to that of a 2-form. It turns out that all symplectic manifolds are also Poisson via the following. Let $f: M\to\mathbb{R}$ be a smooth function on the symplectic manifold (M,ω). Define the vector field X_f through the following procedure:

$$\Omega^1(M)\ni df = \omega(X_f,\cdot),$$

i.e. $X_f = \omega^\sharp(df)$. Then $\{f,g\} = \omega(X_f,X_g)$ is a Poisson bracket. If $M = T^*Q$ is a cotangent bundle with coordinates (x,p), the bracket becomes

$$\{f,g\} = \omega(X_f,X_g) = \sum_i \frac{\partial f}{\partial x^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial x^i}.$$

All symplectic manifolds are Poisson, is the converse true?

Exercise 10. *Let $P=\mathbb{R}^3$ with $q=(x,y,z)$ and consider the bracket*

$$\begin{aligned}\{f,g\}(q) &= \langle q, \nabla f(q)\times\nabla g(q)\rangle\\ &= x\left(\frac{\partial f}{\partial y}\frac{\partial g}{\partial z} - \frac{\partial f}{\partial z}\frac{\partial g}{\partial y}\right) - y\left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial z} - \frac{\partial f}{\partial z}\frac{\partial g}{\partial x}\right) + z\left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}\right).\end{aligned}$$

Prove that this is a Poisson bracket. Here we have an odd dimensional Poisson manifold while all symplectic manifolds must be even dimensional. Conclude that $\mathbb{R}^3$ cannot be symplectic and provide a reasoning (other than being odd dimensional).

For a given non-degenerate 2-form $\alpha\in\Omega^2(M)$, we can construct an almost Poisson bracket by the procedure

$$\{f,g\}_\alpha := \alpha(X_f^\alpha, X_g^\alpha),\quad df = \alpha(X_f^\alpha,\cdot).$$

This bracket is skew, bi-linear, and satisfies Liebniz' rule. However, it does not necessarily satisfy the Jacobi identity.

Exercise 11. *(This is quite challenging) Let f, g, h be smooth functions and define the Jacobiator as*

$$\mathrm{Jac}(f,g,h) = \{f,\{g,h\}_\alpha\}_\alpha + \{g,\{h,f\}_\alpha\}_\alpha + \{h,\{f,g\}_\alpha\}_\alpha.$$

Therefore the bracket is a Poisson bracket if and only if $\mathrm{Jac}(f,g,h) = 0$ for any choice of functions. Prove that

$$\mathrm{Jac}(f,g,h) = d\alpha\left(X_f^\alpha, X_g^\alpha, X_h^\alpha\right)$$

and conclude that a non-degenerate 2-form generates a Poisson bracket if and only if it is closed.

3 Hamiltonian Systems

Let (M,ω) be a symplectic manifold and $H : M \to \mathbb{R}$ be a smooth function. The triple (M,ω,H) forms a Hamiltonian system and generates the Hamiltonian vector field via

$$dH = \omega(X_H, \cdot). \tag{5}$$

Example 14. *Let $M = \mathbb{R}^2$ with coordinates (x,p) and let $\omega = dx \wedge dp$. We have*

$$dH = \frac{\partial H}{\partial x}dx + \frac{\partial H}{\partial p}dp,$$

$$X_H = A\frac{\partial}{\partial x} + B\frac{\partial}{\partial p}, \quad \omega(X_H,\cdot) = Adp - Bdx.$$

Equating these, we see that

$$A = \frac{\partial H}{\partial p}, \quad B = -\frac{\partial H}{\partial x}.$$

This is precisely Hamilton's equations of motion:

$$\dot{x} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial x}.$$

It is also possible to view the dynamics from a functional point of view to utilize the Poisson bracket. Let φ_t be the flow of Hamilton's ordinary differential equations. Then

$$\dot{f} = \left.\frac{d}{dt}\right|_{t=0} f\left(\varphi_t(x)\right) = df(X_H) = \langle \nabla f, X_H\rangle = \{f,H\}$$

Exercise 12. *Consider the Poisson bracket on $\mathbb{R}^3$ given in a previous exercise. Let $g : \mathbb{R} \to \mathbb{R}$ be an arbitrary smooth function and consider the function $f(q) := g\left(\|q\|^2\right)$. Prove that*

$$\{f,H\} = 0,$$

for any smooth function $H : \mathbb{R}^3 \to \mathbb{R}$. Such a function is called a Casimir.

There are two key properties of a Hamiltonian system: energy and volume conservation. Volume is more intricate, but energy is very straightforward.

Proposition 15. $\dot{H} = 0$.

Proof. This follows from the fact that the Poisson bracket is skew:

$$\dot{H} = \{H, H\} = 0.$$

□

Let $M = T^*Q$ be the usual cotangent bundle with coordinates (x, p) and the canonical symplectic form

$$\omega = \sum_i dx^i \wedge dp_i.$$

Then it is associated with the skew-symmetric matrix J, such that $\omega(u, v) = \langle u, Jv \rangle$, and

$$J = \begin{bmatrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{bmatrix}.$$

An alternate way to write $dH = \omega(X_H, \cdot)$ is

$$X_H = J \nabla H. \tag{6}$$

Sometimes, the Hamiltonian vector field is referred to as the *symplectic gradient*. To see that (6) produces the usual answer, we compute

$$\begin{bmatrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{bmatrix} \cdot \begin{bmatrix} H_x \\ H_p \end{bmatrix} = \begin{bmatrix} H_p \\ -H_x \end{bmatrix}$$

A similar procedure can produce Hamiltonian systems in a Poisson manifold. If q is a coordinate, then $\dot{q} = \{q, H\}$. In the $\mathbb{R}^3$ example, we have

$$\begin{aligned} \dot{x} &= \{x, H\} = -y \cdot \frac{\partial H}{\partial z} + z \cdot \frac{\partial H}{\partial y} \\ \dot{y} &= \{y, H\} = x \cdot \frac{\partial H}{\partial z} - z \cdot \frac{\partial H}{\partial x} \\ \dot{z} &= \{z, H\} = -x \cdot \frac{\partial H}{\partial y} + y \cdot \frac{\partial H}{\partial x} \end{aligned}$$

Collapsing this into vector notation with $q = (x, y, z)$, we have

$$\dot{q} = \nabla H \times q.$$

We end with a couple exercises on basic dynamic properties of Hamiltonian systems.

Exercise 13. *Consider a Hamiltonian system*

$$\dot{x} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial x}.$$

Let (x_0, p_0) be a fixed point of the dynamics and let A be the linearization of the dynamics about this point. Prove that if λ is an eigenvalue of A, so is $-\lambda$, $\bar{\lambda}$, and $-\bar{\lambda}$. Conclude that fixed points of Hamiltonian systems cannot be (exponentially) asymptotically stable.

Exercise 14. *(Moderately challenging) Let (x_0, p_0) be a fixed point of a Hamiltonian system. Suppose that it is a hyperbolic fixed point (no eigenvalues have real part zero). Its stable manifold is given by*

$$W^s(x_0, p_0) = \left\{ (x, p) : \lim_{t \to \infty} (x(t), p(t)) = (x_0, p_0) \right\}.$$

Prove that its stable manifold is a Lagrangian submanifold. A submanifold is Lagrangian if its tangent space is a Lagrangian subspace at each point.

References

[1] R. Abraham and J.E. Marsden. *Foundations of Mechanics*. AMS Chelsea publishing. AMS Chelsea Pub./American Mathematical Society, 2008.

[2] V.I. Arnold. *Mathematical Methods of Classical Mechanics*. Graduate Texts in Mathematics. Springer, 1989.

[3] R. Berndt. *An Introduction to Symplectic Geometry*. Graduate studies in mathematics. American Mathematical Society, 2001.