Primer on Invariants

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The contents of these notes will be a nonstandard brief survey of invariants of dynamical systems. To get the most out of these, a strong analysis and differential geometry background will be helpful (measure theory and Cartan's calculus). I recommend that at least one person can do each exercise and that everyone can understand. Below are some dynamics references.

1. Two standard books in the field of nonlinear dynamics are the books by Guckenheimer and Holmes [1], and by Perko [4].
2. The monumental book by Katok and Hassleblatt [2] contains everything but is intimidating.
3. My personal favorite is the book by Zehnder [6].

Throughout, some basics from ordinary differential equations are assumed (e.g. existence/uniqueness, stability). If you are unclear about anything, feel free to ask.

1 Invariant Functions

Consider an ordinary differential equation

$$\dot{x} = X(x), \quad x(0) = x_0, \tag{1}$$

where $x \in M$ can be thought of either a manifold or an open set within $\mathbb{R}^n$ and X will be a smooth vector-field. (Technically, smoothness of the vector-field is overkill as it needs only be Lipschitz.) The solution will be given by its flow:

$$\begin{aligned} \varphi : \mathbb{R} \times M &\to M, \\ (t, x) &\mapsto \varphi(t, x) =: \varphi_t(x). \end{aligned} \tag{2}$$

Strictly speaking, the flow (2) will not exist for all time. However, if M is compact solutions will always exist for all time.

Exercise 1. *Find an example of $\dot{x} = X(x)$ for a smooth X such that solutions do not exist for all time.*

An invariant function is a function which does not change under the flow.

Definition 1. *A function $f : M \to \mathbb{R}$ is an invariant if*

$$f(\varphi_t(x)) = f(x), \tag{3}$$

for all $x \in M$ and $t \in \mathbb{R}$.

The condition (3) can be differentiated to produce the following transport PDE

$$\dot{f} = df_x(X(x)) = \sum_{i=1}^{n} X^i(x)\frac{\partial f}{\partial x^i} = 0. \tag{4}$$

The transport equation (4) can possess many nonconstant solutions. One test to show that only constant solutions exist is the following.

Exercise 2. *Let $x_0 \in M$ such that its forward orbit*

$$\mathcal{O}^+(x_0) = \left\{\varphi_t(x_0) : t \in \mathbb{R}^+\right\}$$

is dense in M. Then if f is a continuous invariant function, it must be constant.

Exercise 3. *For the following systems, does there exist a continuous invariant function? If yes, can you find them? What do their phase portraits look like?*

1. $\dot{x} = y$, $\dot{y} = -\alpha x - \beta x^3$ *for some constants* $\alpha, \beta > 0$.
2. $\dot{x} = x^2 - 1$, $\dot{y} = -y$.

1.1 Sub(?)-Invariant Functions

Rather than being constant along trajectories, suppose that a function is decreasing,

$$f(\varphi_t(x)) \leq f(x).$$

Assuming that f is differentiable, the resulting PDE is

$$\sum_{i=1}^{n} X^i(x)\frac{\partial f}{\partial x^i} \leq 0.$$

Definition 2. *Consider a differential equation $\dot{x} = X(x)$ with a fixed point x_0. A continuously differentiable function $V : M \to \mathbb{R}$ is called a Lyapunov function if*

1. $V(x_0) = 0$ *and* $V(x) > 0$ *whenever* $x \neq x_0$,
2. $\dot{V}(x) \leq 0$.

The existence of a Lyapunov function provides sufficient criteria for stability.

Theorem 3 (Lyapunov Stability)**.** *Suppose that $X(x_0) = 0$. If there exists a Lyaupunov function V, then x_0 is stable. If $\dot{V} < 0$ then x_0 is asymptotically stable.*

The fact that the Lyapunov function has a (global) minima at the fixed point is not important; the important feature is that its value is decreasing. This is displayed by LaSalle's invariance principle.

Theorem 4 (LaSalle Invariance)**.** *Consider the dynamical system $\dot{x} = X(x)$ and let $\Omega \subset M$ be a compact subset that is (positively) invariant under the flow ($\varphi_t(\Omega) \subset \Omega$). Let $V : \Omega \to \mathbb{R}$ be C^1 such that*

$$\dot{V}(x) = \sum_{i=1}^{n} X^i(x) \frac{\partial V}{\partial x^i} \leq 0.$$

Let $S \subset \Omega$ be the largest invariant set where $\dot{V} = 0$. Then every solution with initial point in Ω tends asymptotically to S as $t \to \infty$.

If $S = \{x_0\}$ is a single point, then LaSalle's invariance principle reduces to Lyapunov's theorem.

If a Lyapunov function exists, then the system exhibits asymptotic behavior. Is the converse true? Do asymptotic systems admit a Lyapunov function? The answer is, amazingly, yes!

Theorem 5 (Conley's Fundamental Theorem)**.** *A flow φ_t on a compact metric space has a Lyapunov function V such that*

$$V\left(\varphi_t(x)\right) < V(x), \quad t > 0$$

when $x \notin \mathcal{R}(\varphi_t)$. Additionally, $V\left(\mathcal{R}(\varphi_t)\right)$ is nowhere dense in $\mathbb{R}$.

Here, the set $\mathcal{R}(\varphi_t)$ is the *chain-recurrent* set. This can be informally thought of as a generalization of periodic points. We will not deal with this.

2 Invariant Measures

Let (M, Σ, μ) be a probability Borel measure space. This means that Σ is the smallest σ-algebra containing the open sets and $\mu(M) = 1$. For more on measure theory, see e.g. [5]. However, the level of detail presented there will be overkill.

Definition 6 (Measure Space)**.** *A triple (M, Σ, μ) is a measure space if*

1. *Σ is a σ-algebra over M. This means that Σ is a set of subsets of M such that*
   - *(a) $M \in \Sigma$ (universal set),*
   - *(b) If $A \in \Sigma$, then $M \setminus A \in \Sigma$ (closed under complements), and*

(c) *For a countable collection* $\{A_k\}_{k=1}^{\infty} \in \Sigma$*, then so is* $A = A_1 \cup A_2 \cup \dots$ *(closed under countable unions).*

2. μ *is a measure on* (M, Σ)*. This means that* $\mu : \Sigma \to [0, +\infty]$ *is a function such that*

   (a) *For* $A \in \Sigma$*,* $\mu(A) \geq 0$*,*

   (b) $\mu(\emptyset) = 0$*,*

   (c) *For a countable collection* $\{A_k\}_{k=1}^{\infty} \in \Sigma$ *of pairwise disjoint sets,*

$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \mu(A_k).$$

Everywhere onward, the σ-algebra in question will exclusively be the Borel algebra - the smallest σ-algebra containing the open sets. As such, explicit mention of this collection will be omitted.

We begin with what it means for a measure to be invariant under a dynamical system. Unlike the previous section, the results here will be written from the perspective of discrete dynamics. To modify to continuous dynamics, most of the sums can reasonably be replaced by integrals.

Definition 7. *A map* $T : M \to M$ *is measure-preserving if* $\mu(T^{-1}(A)) = \mu(A)$ *for every measurable set* $A \in \Sigma$*.*

Example 8. *Consider the interval* $M = [0, 1]$ *with the map* $T(x) = 2x \mod 1$*. This map preserves the standard Lebesgue measure. For a pictoral view, see Figure 1. The motivation behind requiring that* $\mu(T^{-1}(A)) = \mu(A)$ *rather than* $\mu(T(A)) = \mu(A)$ *is to account for the fact that the pre-image produces two components and we want to count both. Another way of seeing this is that the doubling map preserves integrals in the following sense:*

$$\begin{aligned} \int_0^1 f\,(Tx)\;dx &= \int_0^{1/2} f(2x)\,dx + \int_{1/2}^1 f(2x-1)\,dx \\ &= \frac{1}{2}\int_0^1 f(y)\,dy + \frac{1}{2}\int_0^1 f(z)\,dz \\ &= \int_0^1 f(x)\,dx, \end{aligned}$$

where the substitutions $y = 2x$ *and* $z = 2x - 1$ *are used.*

An important and natural question to ask is: do invariant measures exist? Clearly the zero measure is always preserved (similar in spirit to the fact that constant functions are always preserved). It turns out that when the space is compact, a non-trivial invariant measure exists.

Theorem 9 (Krylov-Bogolubov)**.** *Any continuous map on a metrizable compact space has an invariant Borel probability measure.*

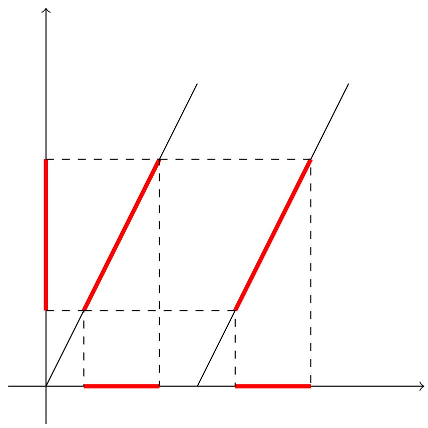

Figure 1: The map $T(x) = 2x \mod 1$. The length of the interval on the y-axis is equal to the sum of the lengths of the intervals on the x-axis. This demonstrates the pre-images have the same measure.

Proof. Let $T : M \to M$ and $x \in M$. Take a dense countable set $\{\varphi_k\}_{k=1}^{\infty}$ in the space $C(M)$ (this space is separable with respect to the supremum norm). For each m, the sequence of numbers

$$\frac{1}{n} \sum_{k=0}^{n-1} \varphi_m \left(T^k(x)\right),$$

is bounded (why?) and hence contains a convergent subsequence via Bolzano-Weierstrass. By a diagonal process, we can refine this subsequence to be convergent for each m,

$$J(\varphi_m) := \lim_{\ell \to \infty} \frac{1}{n_\ell} \sum_{k=0}^{n_\ell - 1} \varphi_m \left(T^k(x)\right).$$

We currently have a way to assign meaning for this sum to the dense collection of continuous functions. What remains to show is that we can do this for an arbitrary continuous function. Let $\varphi \in C(M)$ be arbitrary. For a fixed $\varepsilon > 0$ choose a φ_m such that

$$\sup_{x \in M} \|\varphi(x) - \varphi_m(x)\| < \varepsilon.$$

Then,

$$\frac{1}{n_\ell} \sum_{k=0}^{n_\ell - 1} \varphi \left(T^k(x)\right) = \frac{1}{n_\ell} \sum_{k=0}^{n_\ell - 1} \varphi_m \left(T^k(x)\right) + \frac{1}{n_\ell} \sum_{k=0}^{n_\ell - 1} \left(\varphi\left(T^k(x)\right) - \varphi_m\left(T^k(x)\right)\right).$$

The first term on the right converges to $J(\varphi_m)$ while the second is bounded by ε. Since ε is arbitrary, the limit exists

$$J(\varphi) = \lim_{\ell\to\infty} \frac{1}{n_\ell} \sum_{k=0}^{n_\ell - 1} \varphi\left(T^k(x)\right).$$

This is a linear, bounded, invariant, and positive functional $J : C(M) \to \mathbb{R}$. Therefore,

$$J(\varphi) = \int \varphi \, d\mu_x,$$

by the Riesz representation theorem where μ_x is a T-invariant Borel probability measure. $\square$

It turns out that if we can find an invariant measure (which always exists), then we can learn much about the recurrence properties of the system.

Theorem 10 (Poincaré Recurrence)**.** *Let (M, Σ, μ) be a finite measure space and $T : M \to M$ a measure-preserving map. For a measurable set $A \in \Sigma$, we define the subset $A_0 \subset A$ to be*

$$A_0 := \left\{x \in A : T^k(x) \in A \textit{ for infinitely many } k\right\}.$$

Then A_0 is measurable and $\mu(A_0) = \mu(A)$.

Proof. For a positive integer, $n \geq 1$, define the sets

$$C_n := \left\{x \in A : T^k(x) \notin A, \quad \forall k \geq n\right\}, \quad A_0 = A \setminus \bigcup_{n\geq 1} C_n.$$

The theorem will follow if we can show that every C_n is a measurable null set. As A is a measurable set and T is a measurable map, we see that

$$C_n = A \setminus \bigcup_{j\geq n} T^{-j}(A)$$

is measurable. Moreover,

$$C_n \subset \bigcup_{j\geq 0} T^{-j}(A) \setminus \bigcup_{j\geq n} T^{-j}(A),$$

from which we conclude (which requires that $\mu(M) < \infty$),

$$\mu(C_n) \leq \mu\left(\bigcup_{j\geq 0} T^{-j}(A)\right) - \mu\left(\bigcup_{j\geq n} T^{-j}(A)\right).$$

Since T is measure-preserving, both unions on the right hand side have the same measure and thus $\mu(C_n) = 0$. $\square$

Exercise 4. *Explain how the measure-preserving map $x \mapsto x+1$ on $\mathbb{R}$ does not violate the Poincaré recurrence theorem.*

Exercise 5. *Consider the map $T(x) = \alpha \cdot x$ where $\alpha < 1$. Show that δ_0 (the Dirac measure at the origin) is an invariant probability measure and that we have Poincaré recurrence. Explain how recurrence is strange when the measure is singular.*

2.1 Ergodicity

The main objective of ergodic theory is to understand the following limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} g\left(T^n(x)\right), \quad \text{or} \quad \lim_{n\to\infty} \frac{1}{T} \int_0^T g\left(\varphi_t(x)\right)\, dt, \tag{5}$$

depending on whether we are dealing with a discrete or continuous-time system. This is the measure-theoretic counterpart to the ω-limit set. The first question is: does (5) converge? While the second question is: what does (5) converge to? Convergence is handled by the ergodic theorems while the evaluation can be handled if the system is ergodic. We present one of the ergodic theorems below.

Theorem 11 (Birkhoff's Ergodic Theorem)**.** *Let (M, Σ, μ) be a finite measure space and $T : M \to M$ a measure-preserving map. For every integrable function $f \in L^1(M, \Sigma, \mu)$, there exists a function $f^* \in L^1(M, \Sigma, \mu)$ and a null set $N \subset M$ (depending on f) satisfying $T^{-1}(N) = N$ and*

1. $\displaystyle\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k(x)) = f^*(x)$ *for all* $x \in M \setminus N$
2. $f^*(T(x)) = f^*(x)$ *for all* $x \in M$,
3. $\displaystyle\int_M f^* = \int_M f$,
4. $\displaystyle\int_M \left| \frac{1}{n} \sum_{k=0}^{n-1} f(T^k(x)) - f^*(x) \right| d\mu \to 0$ *as* $n \to \infty$.

This is quite difficult to prove and we will not do so here. Unpacking the results, we see that

1. The ergodic sum (5) converges point-wise almost everywhere,
2. The resulting limiting function is "invariant" (notice the lack of an inverse),
3. The limiting function has the same area, and
4. The ergodic sum converges in the L^1 sense.

(It is also true that the ergodic sum converges in the L^2 sense; this is the von Neumann ergodic theorem.)

Definition 12 (Ergodicity)**.** *Let (M, Σ, μ) be a measure space. A measure-preserving map $T : M \to M$ is ergodic if every measurable set $A \in \Sigma$ that is invariant, $T^{-1}(A) = A$, is either a null set or has full measure, i.e.*

$$\mu(A) = 0, \quad \textit{or} \quad \mu(A) = \mu(M).$$

Exercise 6. *Prove that if a system is ergodic, then the only measurable invariant functions are constant (almost everywhere). How does this compare to the transitive case? Examine the case $T(x) = \alpha \cdot x$ with $\alpha < 1$ and $\mu = \delta_0$.*

We deduce that the ergodic sums must converge (almost everywhere) to a constant function, i.e. it is (almost always) independent of the initial condition!

2.2 Frobenius-Perron

For a given dynamical system, we wish to find an (ergodic) invariant measure. It turns out that we can turn a dynamical system into a new dynamical system on measures (technically densities) such that fixed points under this new system correspond to invariant measures of the original system. This procedure relies on the Frobenius-Perron operator. See [3] for more details on this subject, specifically chapters 3 and 4.

Definition 13. *A measurable map $T : M \to M$ is nonsingular if $\mu(T^{-1}(A)) = 0$ for all measurable sets with $\mu(A) = 0$.*

In particular, if T preserves μ, then is must be nonsingular. If the transformation is nonsingular, the Frobenius-Perron operator can be defined as below.

Definition 14. *The Frobenius-Perron operator $P : L^1 \to L^1$ is defined as assigning the unique function Pf to f such that*

$$\int_A Pf \, d\mu = \int_{T^{-1}(A)} f \, d\mu.$$

for all measurable sets A.

It is important to point out that P preserves measure,

$$\int_M Pf \, d\mu = \int_M f \, d\mu.$$

Example 15. *Consider the map $T(x) = \sin(\pi x)$ on the interval $[0, 1]$. Let be* *the interval* $A = [0, x]$. *Then*

$$\begin{aligned}
Pf(x) &= \frac{d}{dx}\int_{[0,x]} Pf(s)\,ds = \frac{d}{dx}\int_{T^{-1}[0,x]} f(s)\,ds \\
&= \frac{d}{dx}\left[\int_0^{\frac{1}{\pi}\arcsin(x)} f(s)\,ds + \int_{1-\frac{1}{\pi}\arcsin(x)}^{1} f(s)\,ds\right] \\
&= \frac{d}{dx}\left[F(1) - F\left(1-\frac{1}{\pi}\arcsin(x)\right) + F\left(\frac{1}{\pi}\arcsin(x)\right) - F(0)\right] \\
&= \frac{1}{\pi\sqrt{1-x^2}}\left[f\left(1-\frac{1}{\pi}\arcsin(x)\right) + f\left(\frac{1}{\pi}\arcsin(x)\right)\right].
\end{aligned}$$

Theorem 16. *Let* T *be an invertible nonsingular transformation. Then for every* $f \in L^1$,

$$Pf(x) = f\left(T^{-1}(x)\right) J^{-1}(x),$$

where J *is the determinant of the Jacobian matrix.*

The Frobenius-Perron operator provides a test for whether or not a measure is invaraint.

Theorem 17. *Let* $f \geq 0$ *and* $\int f = 1$. *The measure* μ_f *given by*

$$\mu_f(A) = \int_A f\,d\mu,$$

is invariant if and only if $Pf = f$.

Theorem 18. *Let* $T : M \to M$ *be a measure-preserving transformation and* P *the Frobenius-Perron operator. Then*

1. (T, μ) *is ergodic if and only if* $P^n f \stackrel{C}{\to} 1$ *(Cesàro convergent)*
2. (T, μ) *is mixing if and only if* $P^n f \rightharpoonup 1$ *(weakly convergent)*
3. (T, μ) *is exact if and only if* $P^n f \to 1$ *(strongly convergent)*

for any f *such that* $f \geq 0$ *and* $\int f = 1$.

Although we haven't discussed what mixing nor exact means, they both imply ergodic.

Example 19. *Again, consider the map* $T(x) = \alpha \cdot x$ *for* $|\alpha| < 1$. *Then the Frobenius-Perron operator is*

$$Pf(x) = \frac{1}{\alpha} f\left(\frac{x}{\alpha}\right)$$

Upon iterating this map, we have

$$P^n f(x) = \frac{1}{\alpha^n} f\left(\frac{x}{\alpha^n}\right) \rightharpoonup \delta_0,$$

provided that $\int f = 1$.

3 Invariant Forms

In what follows, we specialize the above cases (invariant functions and measures) to the smooth case. A smooth function is a 0-form while a smooth measure (henceforth called a volume) is an n-form (when the space is n-dimensional).

Suppose that $\psi : M \to M$ is a diffeomorphism. A differential k-form $\alpha \in \Omega^k(M)$ is invariant if

$$\psi^*\alpha = \alpha. \tag{6}$$

The notation $\psi^*\alpha$ means the *pullback* of α under ψ. Roughly, this means right composition. Explicitly, let $d\psi$ be the Jacobian matrix of ψ, then

$$\psi^*\alpha_x(v_1, \dots, v_k) = \alpha_{\psi(x)}(d\psi \cdot v_1, \dots, d\psi \cdot v_k).$$

Example 20. *Let $M = \mathbb{R}^2$. We will see what* (6) *means for 0, 1, and 2-forms.*

1. *0-forms are simply smooth functions. Let f be a function, then*

$$\psi^* f(x) = f(\psi(x)).$$

This reduces precisely to (3)*.*

2. *A general 1-form on $\mathbb{R}^2$ has the form*

$$\alpha = f(x, y)dx + g(x, y)dy.$$

Let the map ψ have components

$$\psi(x, y) = \begin{bmatrix} \psi_1(x, y) \\ \psi_2(x, y) \end{bmatrix}$$

Then

$$\begin{aligned} \psi^*\alpha &= (\psi^* f)(d\psi^* x) + (\psi^* g)(d\psi^* y) \\ &= f\left(\psi_1(x, y), \psi_2(x, y)\right) d\psi_1(x, y) + g\left(\psi_1(x, y), \psi_2(x, y)\right) d\psi_2(x, y) \\ &= f(\psi_1(x, y), \psi_2(x, y)) \left[\frac{\partial \psi_1}{\partial x} dx + \frac{\partial \psi_1}{\partial y} dy\right] \\ &\qquad + g\left(\psi_1(x, y), \psi_2(x, y)\right) \left[\frac{\partial \psi_2}{\partial x} dx + \frac{\partial \psi_2}{\partial y} dy\right] \end{aligned}$$

Combining terms, we have $\psi^*\alpha = \alpha$ if

$$\begin{aligned} f\left(\psi_1(x, y), \psi_2(x, y)\right) \frac{\partial \psi_1}{\partial x} + g\left(\psi_1(x, y), \psi_2(x, y)\right) \frac{\partial \psi_2}{\partial x} &= f(x, y) \\ f\left(\psi_1(x, y), \psi_2(x, y)\right) \frac{\partial \psi_1}{\partial y} + g\left(\psi_1(x, y), \psi_2(x, y)\right) \frac{\partial \psi_2}{\partial y} &= g(x, y) \end{aligned}$$

3. *A general 2-form on $\mathbb{R}^2$ has the form*

$$\beta = h(x, y)dx \wedge dy.$$

Following a similar computation, we have

$$\begin{aligned}\psi^*\beta &= (\psi^* h)\,(d\psi^* x) \wedge (d\psi^* y) \\ &= h\left(\psi_1(x,y), \psi_2(x,y)\right)\left[\frac{\partial \psi_1}{\partial x}dx + \frac{\partial \psi_1}{\partial y}dy\right] \wedge \left[\frac{\partial \psi_2}{\partial x}dx + \frac{\partial \psi_2}{\partial y}dy\right] \\ &= h\left(\psi_1(x,y), \psi_2(x,y)\right)\left[\frac{\partial \psi_1}{\partial x}\frac{\partial \psi_2}{\partial y} - \frac{\partial \psi_2}{\partial x}\frac{\partial \psi_1}{\partial y}\right] dx \wedge dy,\end{aligned}$$

where we used the fact that $dx \wedge dy = -dy \wedge dx$ *and* $dx \wedge dx = dy \wedge dy = 0$. *Therefore, a 2-form is invariant if*

$$h\left(\psi_1(x,y), \psi_2(x,y)\right)\left[\frac{\partial \psi_1}{\partial x}\frac{\partial \psi_2}{\partial y} - \frac{\partial \psi_2}{\partial x}\frac{\partial \psi_1}{\partial y}\right] = h(x,y)$$

Exercise 7. *Consider the case above with* $M = \mathbb{R}^2$. *Let* f *be a function and* $df = f_x dx + f_y dy$ *its derivative. Prove that if* f *is invariant under* ψ *(as a 0-form) then* df *is invariant (as a 1-form). Is the converse true?*

Exercise 8. *Consider the 2-form* $\beta = dx \wedge dy$ *on* $\mathbb{R}^2$. *This is the typical volume-form on the plane. Deduce that* $\psi^*\beta = \beta$ *is the usual conditions needed for a map to be volume-preserving.*

3.1 Flows

Most of what you'll be working with this summer will be continuous-time flows rather than discrete-time maps. If φ_t is the flow of our system, a form is preserved if

$$\varphi_t^*\alpha = \alpha.$$

Differentiating this yields the Lie derivative:

$$\mathcal{L}_X\alpha = \left.\frac{d}{dt}\right|_{t=0} \varphi_t^*\alpha.$$

Thus, α is preserved if and only if $\mathcal{L}_X\alpha = 0$.

Example 21. *Let* f *be a function (a 0-form). Then the Lie derivative is precisely the directional derivative:*

$$\mathcal{L}_X f = \left.\frac{d}{dt}\right|_{t=0} f\left(\varphi_t(x)\right) = \langle \nabla f(x), X(x)\rangle = \sum_{i=1}^{n} X(x)^i \frac{\partial f}{\partial x^i}.$$

To calculate the Lie derivative of a higher degree form, we need to utilize the variational equation. Let Φ_t be the Jacobian matrix of the flow φ_t. Then to compute the Lie derivative, we need to understand

$$\left.\frac{d}{dt}\right|_{t=0} \alpha_{\varphi_t(x)}\left(\Phi_t \cdot v_1, \ldots, \Phi_t \cdot v_k\right),$$

which is quite intimidating. A (relatively) easy way around this is via Cartan's magic formula: $\mathcal{L}_X = di_X + i_X d$. To understand this,

1. d is the external derivative, it turns k-forms into $k+1$-forms.

2. i_X is the interior product, it turns k-forms into $k-1$-forms.

The interior product works by placing the vector X into the first slot of the k-form,

$$i_X\alpha(v_1,\ldots,v_{k-1}) = \alpha(X, v_1,\ldots,v_{k-1}).$$

Example 22. *Consider the vector field on $\mathbb{R}^2$, $\dot{x} = X(x,y)$, $\dot{y} = Y(x,y)$ and let $\alpha = dx\wedge dy$. Then if we call $z = [x;y]$ and the vector field $Z(z)$, we have*

$$i_Z\alpha = X(x,y)dy - Y(x,y)dx.$$

Exercise 9. *Let $w = [x;y;z]$ and W be the vector field consisting of the components $\dot{x} = X(x,y,z)$, $\dot{y} = Y(x,y,z)$, and $\dot{z} = Z(x,y,z)$. Write out $i_W\alpha$ for when α is a generic 1, 2, or 3-form.*

Remark 23. *By convention, if f is a 0-form (so a function), $i_X f = 0$ as -1-forms don't exist.*

We can now utilize Cartan's magic formula to calculate a Lie derivative.

Example 24. *Consider the dynamics $\dot{x} = x+y$ and $\dot{y} = -y$. This has the flow*

$$\varphi_t(x,y) = \begin{bmatrix} e^t(x+y) - e^{-t}y \\ e^{-t}y \end{bmatrix}$$

The Jacobian matrix of the flow is given by

$$J_\varphi = \begin{bmatrix} e^t & e^t - e^{-t} \\ 0 & e^{-t} \end{bmatrix}$$

Let us consider the 2-form $\beta = ydx\wedge dy$. Its pullback is given by

$$\begin{aligned} \varphi_t^*\beta &= e^{-t}y\left[e^t dx + (e^t - e^{-t})dy\right] \wedge \left[e^{-t}dy\right] \\ &= e^{-t}ydx\wedge dy \end{aligned}$$

Thus its Lie derivative is

$$\mathcal{L}_X\beta = \left.\frac{d}{dt}\right|_{t=0} e^{-t}ydx\wedge dy = -ydx\wedge dy$$

Comparing this with Cartan's magic formula yields

$$\begin{aligned} \mathcal{L}_X\beta &= di_X\beta + i_X d\beta \\ &= d\left(y(x+y)dy + y^2dx\right) \\ &= ydx\wedge dy + 2ydy\wedge dx = -ydx\wedge dy. \end{aligned}$$

We get the same answer both ways!

3.2 Divergence

Let M be an n-dimensional manifold (or just $\mathbb{R}^n$). Let μ be an n-form which never vanishes. Such a form is called a *volume* form.

Example 25. *On $\mathbb{R}^3$, the form $dx \wedge dy \wedge dz$ is a volume form but $x^2 dx \wedge dy \wedge dz$ is not as it vanishes when $x = 0$ (on the yz-plane).*

Exercise 10. *Let μ be a volume form and η be an arbitrary n-form. Prove that there exists a (unique) function f such that $\eta = f\mu$.*

In light of the previous exercise, we can define the divergence of a vector field (with respect to a volume form).

Definition 26. *Let X be a vector field and μ a volume form. The unique function f such that*

$$\mathcal{L}_X \mu = f\mu$$

is called the divergence and is written $\operatorname{div}_\mu(X)$.

Exercise 11. *Let $\mu = dx \wedge dy$ be a volume form on $\mathbb{R}^2$. Show that the divergence agrees with the usual definition, i.e. if $z = [x; y]$ and $\dot{x} = X(x, y)$, $\dot{y} = Y(x, y)$ then*

$$\operatorname{div}_\mu(Z) = \frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}.$$

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